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RAN-2203000205023001

Third Year B. Sc. (Sem.-V) Examination October - 2025
Mathematics Paper-XI MTH-501- Group Theory (Old Course)

Time: 2 Hours] **[Total Marks: 50**

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book (2) All questions are compulsory. (3) Figures to the right indicate marks of the questions. (4) Follow usual notations.	Student's Signature

Q.1. Answer the following as directed: (Any FIVE) **(10)**

(1) Prove that the identity element in a group G is unique.
(2) Define: (i) Abelian Group;
(ii) Order of Element in Group.
(3) If G is a finite group and a in G, then $a^{o(G)} = e$.
(4) State Lagrange's Theorem and Euler's Theorem.
(5) Define: (i) Kernel of Homomorphism of Groups;
(ii) Isomorphism of Groups.

(6) Let the mapping $\phi : G \to G$ of a group G be defined by $a\phi = a^{-1}$; for every element a in G. If ϕ is an automorphism of G, then prove that G is abelian.
(7) If $\theta = (1\ 7\ 6\ 4)$ and $\sigma = (5\ 7\ 3) \cdot (6\ 2)$ are the permutations in S_7, then find $\theta^{-1} \cdot \sigma \cdot \theta$.
(8) Which of the following permutations in S_8 are even:
(a) $(5\ 8\ 3) \cdot (4\ 6\ 1\ 7) \cdot (8\ 7\ 2)$;
(b) $(18\ 7\ 4\ 2\ 3) \cdot (8\ 2\ 5\ 6\ 4) \cdot (5\ 8)$.

Q.2. Attempt any TWO: **(10)**

(a) Let G be a group. Prove that G is abelian if and only if $(a \cdot b)^2 = a^2 \cdot b^2$; for all a, b in G.
(b) Prove that a non - empty subset H of a group G is a subgroup of G if and only if (i) $a, b \in H \Rightarrow a \cdot b \in H$;
(ii) $a \in H \Rightarrow a^{-1} \in H$.
(c) (i) Prove that a cyclic group is abelian.
(ii) If for some elements a, b in a group G; $o(a) = 3$ and $a \cdot b \cdot a^{-1} = b^2$, then find $o(b)$.

Q.3. Attempt any TWO: **(10)**

(a) Prove that the relation of "*congruence modulo H*" is an equivalence relation on a group G ; where H is a subgroup of G.
(b) If the order of a finite group G is a prime number p, then prove that G is cyclic.
(c) Let H and K be subgroups of a group G. If $HK = KH$, then prove that HK is a subgroup of G.

Q.4. Attempt any TWO: (10)

(a) Define Normal Subgroup. Give an illustration of a normal subgroup of a non-abelian group. Prove that every subgroup of an abelian group is normal.

(b) Define Homomorphism of Groups. Let $\phi : G \to \overline{G}$ be a homomorphism of a group G in to a group $\overline{G}$. Then prove that:
(i) $\phi(e) = \overline{e}$, where e, $\overline{e}$ are the identity elements of groups G and $\overline{G}$ respectively;
(ii) $\phi(x^{-1}) = [\phi(x)]^{-1}$; for every x in G.

(c) Let $\phi : G \to \overline{G}$ be a homomorphism of a group G in to a group $\overline{G}$; with the *Kernel*; *Ker* (ϕ). Prove that a homomorphism $\phi : G \to \overline{G}$ is an isomorphism if and only if *Ker* $(\phi) = (e)$; where e is the identity element of G.

Q.5. Attempt any TWO: (10)

(a) (i) If $\sigma = (1\ 3\ 5\ 7) \cdot (8\ 6\ 4\ 2)$ the permutations in S_8, then find σ^{-1}.
(ii) Verify whether $\tau^{-1} \cdot \omega \cdot \tau$ is even or odd permutation; for the permutations $\tau = (1\ 2\ 7\ 6)$ and $\omega = (4\ 5\ 2)\cdot(1\ 3)$ in S_7.

(b) Prove that there does not exist any permutation θ in S_9 satisfying

$$\theta^{-1} \cdot (9\ 5\ 3)\cdot\theta = (7\ 4\ 1\ 6).$$

(c) Given the permutations $\sigma = (3\ 4)\cdot(2\ 6)$ and $\tau = (1\ 6)\cdot(5\ 2)$ in S_6; find a permutation θ in S_6 satisfying $\theta^{-1} \cdot \sigma \cdot \theta = \tau$.
